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Thermodynamic costs of temperature stabilization in logically irreversible computation

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Abstract: In recent years, great efforts are devoted to reducing the work cost of the bit operation, but it is still unclear whether these efforts are sufficient for resolving the temperature stabilization problem in computation. By combining information thermodynamics and a generalized constitutive model which can describe Fourier heat conduction as well as non-Fourier heat transport with nonlocal effects, we here unveil two types of the thermodynamic costs in the temperature stabilization problem. Each type imposes an upper bound on the amount of bits operated per unit time per unit volume, which will eventually limit the speed of the bit operation. The first type arises from the first and second laws of thermodynamics, which is independent of the boundary condition and can be circumvented in Fourier heat conduction. The other type is traceable to the third law of thermodynamics, which will vary with the boundary condition and is ineluctable in Fourier heat conduction. These thermodynamic costs show that reducing the work cost of the bit operation is insufficient for resolving the temperature stabilization problem in computation unless the work cost vanishes.

Keywords: computational system; temperature stabilization; bit operation; heat transport

1 Introduction

For a practical computational system, stabilizing its operating temperature at a satisfactory level is important because the operating temperature can strongly influence its performance and reliability [1]. Unfortunately, this demand is nowadays challenged by the extremely high heat flux in the computational system, which has limited the computational performance for almost two decades [2], [3]. To face this challenge, great efforts have been devoted to minimizing the work cost of the bit operation [4]–[10], which enables us to lower the heat flux as much as possible. However, recent achievements in the field of information thermodynamics have unveiled the physical limits of such minimizations, i.e., the finite-time Landauer principles [11]–[14]. The finite-time Landauer principles give lower bounds on the work cost of the bit-erasure operation, which take the form

$$W_E = W_L + \sum T \geq \left(k_B \ln 2 + \sum_{\min} \right) T. \tag{1}$$

Here, W_E denotes the work cost per bit erased, $W_L = k_B T \ln 2$ is the celebrated Landauer limit [15]–[23], k_B is the Boltzmann constant, T stands for the absolute temperature of the position where the bit-erasure operation takes place, $\sum$ is the averaged entropy generation due to the finite-time effect, and $\sum_{\min}$ is the minimum of $\sum$, which is typically determined by the system parameters as well as the protocol duration. The condition for

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$\sum = \sum_{\min}$ will show us how to minimize the work cost, but owing to $\sum_{\min} \geq 0$, the minimal work cost is always larger than the Landauer limit. For the purpose of reducing the work cost further, Ray and Crutchfield [5] have proposed a class of the bit-swap operations, whose minimal work cost can reach the sub-Landauer level, namely,

$$\min\{W_S\} = 0.43W_L, \tag{2}$$

where W_S is the work cost per bit swapped.

Although minimizing the work cost as mentioned above will benefit the temperature stabilization without doubt, it is still unclear whether such efforts are sufficient for resolving the temperature stabilization problem in computation. The previous studies restricted to the field of information thermodynamics are not able to answer this question because they have not take into account heat transport. The work cost of the bit operation will dissipate as heat, and in the absence of heat transport, all heat will accumulate in the computational system, which inevitably poses a rise of the operating temperature. Therefore, a consideration on both information thermodynamics and heat transport is necessary for the temperature stabilization problem in computation, which has not been much discussed [24]. We mention that Sciacca and Alvarez have investigated another topics related to non-Fourier heat transport and the concept of information [25].

In the present work, we investigate the one-dimensional and linear temperature stabilization problem which satisfies the following assumptions. First, the bit operation is uniformly distributed and coexists with one-dimensional heat transport on a finite domain $[0, L]$. Second, the local equilibrium is achieved everywhere on this domain, which allows us to define the spatial distribution of the absolute temperature, $T = T(x)$. Third, the work cost of the bit operation can be fitted by

$$W_O = aW_L + b, \tag{3}$$

where W_O is the work cost per bit operated, constant a is strictly positive and constant b is non-negative. The results in Ref. [5] scales as $W_O(a > 0, b = 0)$, and $W_O(a = 1, b > 0)$ can cover the lower bound of the finite-time Landauer principle which goes beyond weak coupling [14]. Finally, heat transport can be modeled in terms of

$$q = -\kappa\nabla T + l_1^2\nabla^2 q + l_2^2\nabla\left(\nabla \cdot q\right), \tag{4}$$

where $q = q(x)$ is the one-dimensional heat flux, κ is the constant thermal conductivity, l_1 and l_2 are non-negative constants. Eq. (4) has been derived from various modeling methods [26]–[35], and in most methods, l_1 and l_2 are associated with the mean free path of the heat carriers. If at least one of l_1 and l_2 is strictly positive, Eq. (4) depicts nonlocal heat transport, while the case of $l_1 = l_2 = 0$ corresponds to conventional Fourier heat conduction. Indeed, Eq. (4) discards the terms with respect to the relaxation times as well as the temporal derivatives of the heat flux and the local temperature, which describe the relaxational effects. That is because we here concentrate on steady-state heat transport.

Based on the modeling assumptions mentioned above, it is theoretically demonstrated that there are two types of constraints in the temperature stabilization problem. The first type arises from the first and second laws of thermodynamics, and the other one is a result of the third law of thermodynamics. If these constraints are not satisfied, the corresponding temperature stabilization problem will be physically meaningless. Such constraints are conceptualized as the thermodynamic costs of the temperature stabilization in computation, and elucidate why minimizing the work cost of the bit operation is insufficient for resolving the temperature stabilization problem in computation.

2 Constraint imposed by the first and second laws of thermodynamics

In the temperature stabilization problem, the first law of thermodynamics is stated as

$$W_O I - \left(\nabla \cdot q\right) = 0, \tag{5}$$

where I is the amount of bits operated per unit time per unit volume. For one-dimensional steady-state heat transport, combining Eqs. (3)–(5) leads to

$$q = -\left[\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2\right]\nabla T. \tag{6}$$

It should be emphasized that Eqs. (5) and (6) are for the three-dimensional system. For the one-dimensional and two-dimensional systems, we should introduce the amount of bits operated per unit time per unit length and per unit area respectively. On the other hand, the second law of thermodynamics enforces that any heat transport process must have a non-negative entropy production [36], namely,

$$\begin{aligned}\sigma &= q \cdot \nabla\left(-T^{-1}\right) \\ &= \left[\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2\right] \cdot \left|\nabla\left(T^{-1}\right)\right|^2 \geq 0,\end{aligned} \tag{7a}$$

where σ is the local entropy production rate of heat transport. Owing to $\left|\nabla\left(T^{-1}\right)\right|^2 \geq 0$, inequality (7a) can be simplified as

$$\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2 \geq 0. \tag{7b}$$

The first and second laws of thermodynamics are mathematically valid if and only if inequality (7b) is satisfied.

For non-Fourier heat transport with the nonlocal effect ($l_1^2 + l_2^2 > 0$), inequality (7b) will not be satisfied unless

$$I \leq I_1 = \frac{\kappa}{a\left(l_1^2 + l_2^2\right)k_B \ln 2}. \tag{8}$$

According to inequality (8), the amount of bits operated per unit time per unit volume must be upper bounded by I_1. Otherwise, the first and second laws of thermodynamics will not be mathematically valid. Such thermodynamic constraint reveals that the temperature stabilization comes at a cost, so it can be conceptualized as the thermodynamic cost of the temperature stabilization in computation. Because the volume of a practical computational system cannot be infinite, this thermodynamic cost will eventually limit the speed of the bit operation. In other words, in the presence of nonlocal heat transport, a excessively fast bit operation will render the temperature stabilization thermodynamically impossible. When it comes to Fourier heat conduction ($l_1 = l_2 = 0$), inequality (7b) is necessarily satisfied for arbitrary $I \in (0, +\infty)$. As a result, the first and second laws of thermodynamics always possess the mathematical validity, and will not limit the speed of the bit operation.

3 Constraint imposed by the third law of thermodynamics

The third law of thermodynamics requires the positive absolute temperature, and its mathematical validity will vary with specific boundary value problems (BVPs). We here concern three commonly used BVPs. The first one possesses the Dirichlet boundary condition as follows,

$$T(x) = \begin{cases} T_1, x = 0 \\ T_2, x = L \end{cases} \tag{9}$$

Because T_1 and T_2 are actually the absolute temperatures of the boundary points, they must be positive. In the second BVP, the boundary heat fluxes are given,

$$q(x) = \begin{cases} q_1, x = 0 \\ q_2, x = L \end{cases} \tag{10a}$$

Note that the boundary heat fluxes are for cooling the computational system. Accordingly, q_1 and q_2 should be non-positive and non-negative respectively. By the virtue of Eq. (6), Eq. (10a) can be reformulated as the standard Neumann boundary condition

$$\nabla T(x) = \begin{cases} -\dfrac{q_1}{\kappa_{eff} - aI\left(l_1^2 + l_2^2\right)k_B \ln 2}, x = 0 \\ -\dfrac{q_2}{\kappa_{eff} - aI\left(l_1^2 + l_2^2\right)k_B \ln 2}, x = L \end{cases} \tag{10b}$$

The third BVP describes the scenario wherein the one surface of the computational system is adiabatic, and the other surface is cooled by a heat sink. Without loss of generality, the boundary condition of such BVP can be written as

$$q(x) = \begin{cases} 0, x = 0 \\ R_0^{-1}\left(T - T_0\right), x = L \end{cases}, \tag{11a}$$

where $R_0 > 0$ stands for the thermal boundary resistance between the computational system and the heat sink [37]–[40], and $T_0 > 0$ is the absolute temperature of the heat sink. Similarly, Eq. (11a) can also be reformulated as a mix of the standard Neumann and Robin boundary conditions, namely,

$$\left[\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2\right]\nabla T(x) = \begin{cases} 0, x = 0 \\ R_0^{-1}\left(T_0 - T\right), x = L \end{cases} \tag{11b}$$

In order to avoid the nonlinearity, all parameters in the aforementioned boundary conditions are assumed to be temperature-independent.

Combining Eqs. (4)–(6) yields

$$\nabla^2 T + \frac{aIk_B \ln 2}{\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2} T + \frac{bI}{\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2} = 0. \tag{12}$$

The general solution of Eq. (12) takes the form

$$T = C_1 \sin\left(\frac{\alpha x}{L}\right) + C_2 \cos\left(\frac{\alpha x}{L}\right) - \frac{b}{ak_B \ln 2}, \alpha = \sqrt{\frac{aIk_B L^2 \ln 2}{\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2}}. \tag{13}$$

where C_1 and C_2 are the undetermined coefficients. For this solution, the third law of thermodynamics is mathematically valid if and only if

$$y(\alpha) = \min_{0 \le x \le L}\left\{C_1 \sin\left(\frac{\alpha x}{L}\right) + C_2 \cos\left(\frac{\alpha x}{L}\right)\right\} > \frac{b}{ak_B \ln 2}. \tag{14}$$

Note that $y(\alpha)$ is always non-positive for any $\alpha \ge \pi$, which is contradictory to inequality (14). As a consequence, inequality (14) has the following the necessary condition,

$$\alpha < \alpha_1 = \pi, \tag{15a}$$

which is equivalent to

$$I < I_2 = \frac{\kappa}{a\left(l_1^2 + l_2^2 + L^2\pi^{-2}\right)k_B \ln 2}. \tag{15b}$$

Similar to inequality (8), inequality (15b) also imposes an upper bound on the amount of bits operated per unit time per unit volume, which will not vary with the boundary values. Nonetheless, we cannot yet regard inequality (15b) as another thermodynamic cost of the temperature stabilization in computation. That is because the sufficient condition for inequality (14) is still lacking, which will be presented in the following.

With inequality (15b) satisfied, inequality (14) can be simplified as

$$y(\alpha) = \min\{C_2, C_1 \sin \alpha + C_2 \cos \alpha\} > \frac{b}{ak_B \ln 2}. \tag{16}$$

For the first BVP, the undetermined coefficients are obtained as

$$\begin{cases} C_2 = \dfrac{T_2 - T_1 \cos \alpha}{\sin \alpha} + \dfrac{b}{ak_B \ln 2} \tan \dfrac{\alpha}{2} \\ C_2 = T_1 + \dfrac{b}{ak_B \ln 2} \end{cases} \tag{17}$$

By substituting Eq. (17) in to inequality (16), we can acquire the sufficient and necessary condition for inequality (14), namely,

$$\begin{aligned} y(\alpha) &= y_1(\alpha) \\ &= \min\{T_2, T_1\} + \frac{b}{ak_B \ln 2} > \frac{b}{ak_B \ln 2}, \end{aligned} \tag{18}$$

where function $y_1(\alpha)$ is defined on $(0, \pi)$. Mathematically, inequality (18) always holds, which means that inequality (15b) is the necessary and sufficient condition for inequality (14). Therefore, inequality (15b) is another thermodynamic cost of the temperature stabilization in computation. For a finite computational system, this thermodynamic cost will entail a speed limit to the bit operation as well. Unlike inequality (8), inequality (15b) remains well-defined in the case of $l_1 = l_2 = 0$. It means that in Fourier heat conduction, I_2 is the unique upper bound on I. In the case of nonlocal heat transport ($l_1^2 + l_2^2 > 0$), I_2 is smaller than I_1 as shown below,

$$\frac{I_2}{I_1} = 1 - \frac{L^2\pi^{-2}}{l_1^2 + l_2^2 + L^2\pi^{-2}} < 1, l_1^2 + l_2^2 > 0. \tag{19}$$

These facts imply that for the temperature stabilization problem described by the first BVP, inequality (15b) always plays a dominant role in constraining I as well as limiting the speed of the bit operation.

The coefficients corresponding to the second BVP are given by

$$\begin{cases} C_1 = -\dfrac{q_1 L}{\kappa\alpha - a\alpha I\left(l_1^2 + l_2^2\right)k_B \ln 2} \\ C_2 = \dfrac{q_2 L \csc \alpha - q_1 L \cot \alpha}{\kappa\alpha - a\alpha I\left(l_1^2 + l_2^2\right)k_B \ln 2} \end{cases}, \tag{20}$$

and inequality (14) is then simplified as

$$\begin{aligned} y(\alpha) &= y_2(\alpha) \\ &= \frac{\left(\max\{-q_1, q_2\} \cos \alpha + \min\{-q_1, q_2\}\right)L}{\left[\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2\right]\alpha \sin \alpha} > \frac{b}{ak_B \ln 2}, \end{aligned} \tag{21}$$

where function $y_2(\alpha)$ is defined on $(0, \pi)$. We now prove that in this case, the sufficient and necessary condition for inequality (14) is compose of

$$\max\{-q_1, q_2\} > 0, \tag{22a}$$

and

$$I < I_3 = \frac{\kappa}{a\left(l_1^2 + l_2^2 + L^2\alpha_2^{-2}\right)k_B \ln 2}, \alpha_2 = f\left(y_2 = \frac{b}{ak_B \ln 2}\right), \tag{22b}$$

where $f(y_2)$ denotes the inverse function of $y_2(\alpha)$ and is defined on $(-\infty, +\infty)$. If inequality (22a) is not satisfied, the following corollary necessarily holds,

$$q_1 = q_2 = 0 \Rightarrow y_2(\alpha) \equiv 0 < \frac{b}{ak_B \ln 2}, \tag{23}$$

which is contradictory to inequality (21). Consequently, inequality (22a) is the necessary condition for inequality (14). With inequality (22a) satisfied, we can acquire

$$\lim_{\alpha \to 0^+} y_2(\alpha) = +\infty, \tag{24a}$$

$$\lim_{\alpha \to \pi^-} y_2(\alpha) = -\infty. \tag{24b}$$

Eqs. (24a) and (24b) indicate that if $f(y_2)$ exists, it should be defined on $(-\infty, +\infty)$. On the other hand, the derivative of $y_2(\alpha)$ can be calculated as

$$\frac{dy_2(\alpha)}{d\alpha} = \frac{\left[f_1(\alpha) + f_2(\alpha)\right]L}{\left[\kappa - aI\left(l_1^2 + l_2^2\right)\kappa_B \ln 2\right]\alpha^2 \sin^2 \alpha}, \tag{25}$$

with

$$\begin{cases} f_1(\alpha) = \max\{-q_1, q_2\}\left(\alpha + \cos \alpha \sin \alpha\right) \\ f_2(\alpha) = \min\{-q_1, q_2\}\left(\alpha \cos \alpha + \sin \alpha\right) \end{cases} \tag{26}$$

By utilizing the inequality chain

$$\left(\alpha + \sin \alpha\right) > \left(\alpha + \cos \alpha \sin \alpha\right) > 0, \forall \alpha \in (0, \pi), \tag{27}$$

we can deduce

$$\begin{aligned} f_1(\alpha) + f_2(\alpha) &> \min\{-q_1, q_2\}\left(\alpha + \cos \alpha \sin \alpha\right) + g(\alpha) \\ &= \min\{-q_1, q_2\}\left(\alpha + \sin \alpha\right)(1 + \cos \alpha) > 0 \\ &\Rightarrow \frac{dy_2(\alpha)}{d\alpha} < 0, \end{aligned} \tag{28}$$

and hence, $f(y_2)$ necessarily exists. Meanwhile, inequality (21) is necessarily satisfied as long as $\alpha < \alpha_2$, which is equivalent to inequality (22b). Taking all these into account, the sufficient and necessary condition for inequality (14) is composed of inequalities (22a) and (22b).

Similarly, for the temperature stabilization problem described by the second BVP, the composite of inequalities (22a) and (22b) can be regarded as a thermodynamic cost of the temperature stabilization in computation. The component given by inequality (22a) is a lower bound constraint on the boundary values, and has nothing to do with the amount of bits operated per unit time per unit volume. That is the most main difference between this thermodynamic cost and inequality (8). The physical meaning of inequality (22a) is that the computational system cannot be adiabatic at both boundary points. Otherwise, the temperature stabilization is thermodynamically impossible. This physical meaning can be interpreted from the viewpoint of the energy balance, which is indispensable for the temperature stabilization. If the computational system is adiabatic at both boundary points, the energy balance necessarily requires that the total heat generation is zero. However, according to Eq. (3), the total heat generation is strictly positive unless the negative absolute temperature coexists with the positive absolute temperature. Such coexistence will inevitably violate the third law of thermodynamics, so the corresponding temperature stabilization problem must be thermodynamically impossible. The other component, inequality (22b), is an upper bound constraint on the amount of bits operated per unit time per unit volume. Meanwhile, the upper bound I_3 will exhibit the following mathematical behaviors,

$$I_3 < +\infty, l_1 = l_2 = 0, \tag{29a}$$

$$I_3 < I_1, l_1^2 + l_2^2 > 0. \tag{29b}$$

which mean that for the temperature stabilization problem described by the second BVP, inequality (22b) will always dominate the speed limit to the bit operation.

When it comes to the third BVP, we can acquire

$$\begin{cases} C_1 = 0 \\ C_2 = g(\alpha)\left[T_0 + b/\left(ak_B \ln 2\right)\right] \end{cases}, \tag{30}$$

with

$$g(\alpha) = \frac{L}{L \cos \alpha - R\left[\kappa - aI\left(l_1^2 + l_2^2\right)k_B \ln 2\right]\alpha \sin \alpha}. \tag{31}$$

In this case, inequality (14) becomes

$$\begin{aligned} y(\alpha) &= y_3(\alpha) \\ &= \frac{\cos \alpha}{g(\alpha)}\left(T_0 + \frac{b}{ak_B \ln 2}\right) > \frac{b}{ak_B \ln 2}, \end{aligned} \tag{32}$$

where function $y_3(\alpha)$ is defined on $\left(0, \frac{\pi}{2}\right)$. Using the method stated above, the following thermodynamic cost can be derived,

$$I < I_4 = \frac{\kappa}{a\left(l_1^2 + l_2^2 + L^2\alpha_3^{-2}\right)k_B \ln 2}, \alpha_3 = h\left(y_3 = 0\right), \tag{33}$$

where $h\left(y_3\right)$ is the inverse function of $y_3(\alpha)$ and defined on $(-\infty, +\infty)$. Similarly, inequality (33) is also an upper bound the amount of bits operated per unit time per unit volume. Moreover, inequalities (29a) and (29b) will still hold when I_3 is replaced by I_4. Thus, for the temperature stabilization problem described by the third BVP, the speed limit to the bit operation is dominated by inequality (33).

4 Conclusions

1. The laws of thermodynamics will impose two types of the constraints on the temperature stabilization in computation, which can be considered as the thermodynamic costs of the temperature stabilization in computation. The forms of these thermodynamic costs are strongly influenced by the thermal conductivity, the work cost of the bit operation, and the nonlocal effect of heat transport.
2. The first type of the thermodynamic cost originates from the first and second laws of thermodynamics. This type will not vary with the boundary condition, and can be circumvented in Fourier heat conduction. In nonlocal heat transport, its form is an inequality that imposes an upper bound on the amount of bits operated per unit time per unit volume. For a finite computational system, such upper bound will limit the speed of the bit operation.
3. The other type of the thermodynamic cost is traceable to the third law of thermodynamics, whose form relies on the boundary condition. For the temperature stabilization problem which is described by the Dirichlet or mixed BVP, its form is also an inequality that imposes an upper bound on the amount of bits operated per unit time per unit volume. For the temperature stabilization problem which is described by the Neumann BVP, its forms is compose of an inequality constraining the boundary heat fluxes and an inequality that imposes an upper bound on the amount of bits operated per unit time per unit volume.
4. The latter type of the thermodynamic cost can never be circumvented in Fourier heat conduction, and in nonlocal heat transport, its corresponding upper bound is always smaller than that of the former type. As a consequence, the latter type will play a dominant role in limiting the speed of the bit operation. Furthermore, because the latter type is ineluctable, reducing the work cost of the bit operation is insufficient for resolving the temperature stabilization problem in computation.

5. The main differences between this work and Ref. [24] are as follows. First, this work considers the nonlocality of non-Fourier heat transport, whereas Ref. [24] is restricted to Fourier heat conduction. In the absence of the nonlocality, the first and second laws of thermodynamics can be automatically satisfied, so there is no corresponding constraint in Ref. [24]. Moreover, Ref. [24] actually neglects the thermal boundary resistance between the computational system and the heat sink, whose influence is shown in this work. Finally, this work concentrates on the steady-state temperature field, and does not involve the unsteady-state temperature wave.

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